Higher Gauss sums of modular categories

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- Semisimple, $\mathrm{Irr}(\mathcal{C})$ finite. $\mathcal{C}(X, Y) = \delta_{X,Y}\mathbb{C}$ for $X, Y \in \mathrm{Irr}(\mathcal{C})$. $\mathbb{1} \in \mathrm{Irr}(\mathcal{C})$.

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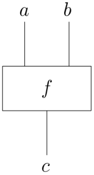
- Fusion rule
$$X \otimes Y = \bigoplus_{Z \in \mathrm{Irr}(\mathcal{C})} N_{XY}^{Z} Z.$$

$K_0(\mathcal{C}) := \mathrm{span}_{\mathbb{C}}\{\mathrm{Irr}(\mathbb{C})\}$ is the Grothendieck ring with multiplication given by $\otimes$.

Graphical calculus on morphisms:



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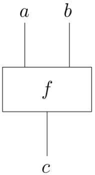

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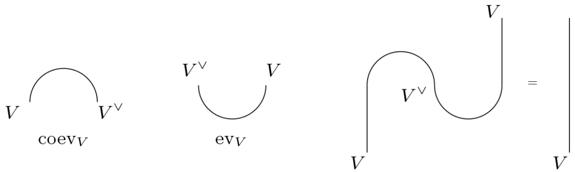
Rigidity of the duality: $(V^\vee, \mathbb{1} \to V \otimes V^\vee, V^\vee \otimes V \to \mathbb{1})$.



$$(\mathrm{id}_V \otimes \mathrm{ev}_V) \circ \alpha_{V,V^\vee,V} \circ (\mathrm{coev}_V \otimes \mathrm{id}_V) = \mathrm{id}_V$$

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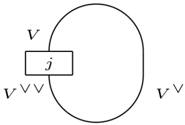
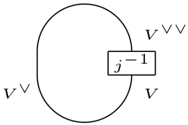
$d_L(V) =$ 

$d_R(V) =$ 

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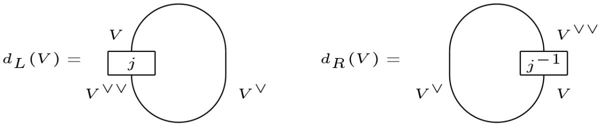

A pivotal structure j of $\mathcal{C}$ is called *spherical* if $d_L(V) = d_R(V)$ for all $V \in \mathcal{C}$.
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Global dimension: $\dim(\mathcal{C}) = \sum_{V \in \mathrm{Irr}(\mathcal{C})} \dim(V)^2$.

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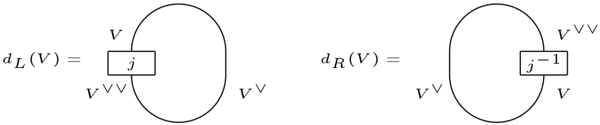

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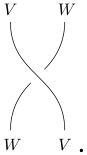
Global dimension: $\dim(\mathcal{C}) = \sum_{V \in \mathrm{Irr}(\mathcal{C})} \dim(V)^2$.
$\mathrm{Tr}_{\mathcal{C}}(f) \in \mathcal{C}(\mathbb{1}, \mathbb{1})$ for $f \in \mathcal{C}(V, V)$: insert f in suitable position.

Braided spherical category

Braiding:

$c_{VW} : V \otimes W \xrightarrow{\cong} W \otimes V$ 

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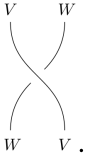
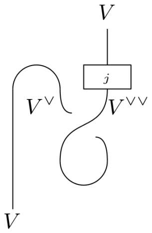


Hexagon axioms.

Twist:

$\theta(V) : V \xrightarrow{\cong} V$



$\theta(V) = \theta_V \operatorname{id}_V, \forall V \in \operatorname{Irr}(\mathcal{C})$.

(Vafa) For all $V \in \operatorname{Irr}(\mathcal{C})$, θ_V is a root of unity.

Modular category

Definition

A braided spherical fusion category is called modular if its S-matrix

$$S_{V,W} = \mathrm{Tr}_{\mathcal{C}}(c_{W,V^\vee} \circ c_{V^\vee,W}),\ V, W \in \mathrm{Irr}(\mathcal{C})$$

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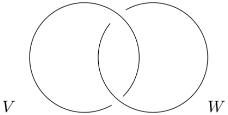

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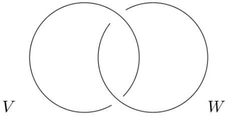

In particular, $S_{\mathbb{1},V} = \dim(V)$, and $S_{V,W} = S_{W,V}$.
Source of examples: finite quadratic modules $\mathcal{C}(G, q)$, $\mathcal{Z}(\mathrm{Rep}(G))$, quantum groups $\mathcal{C}(\mathfrak{g}, k)$ at roots of unity, VOA, subfactor, ...

$T_{V,W} := \theta_V \delta_{V,W}$, $V, W \in \mathrm{Irr}(\mathcal{C})$. In particular, $\mathrm{ord}(T) < \infty$. S- and T-matrices are called the modular data of $\mathcal{C}$.

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Modular category $\Rightarrow \rho : \mathrm{SL}(2, \mathbb{Z}) \to \mathrm{PGL}(K_0)$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto S, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto T.$$

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- They are provided by a (2+1)-TQFT associated to $\mathcal{C}$.

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Quadratic Gauss sum of (G, q): $\tau = \sum_{g \in G} q(g)$. If $|G|$ is odd,

$$\frac{\tau}{\bar{\tau}} = \left(\frac{-1}{|G|}\right) \in \{\pm 1\}$$

Higher Gauss sums and higher central charges

Definition

Let $\mathcal{C}$ be a modular category and $n \in \mathbb{Z}$. The n^{th} Gauss sum of $\mathcal{C}$ is defined as

$$\tau_n(\mathcal{C}) := \sum_{V \in \mathrm{Irr}(\mathcal{C})} \theta_V^n \dim(V)^2.$$

If $\tau_n(\mathcal{C}) \neq 0$, we respectively define the n^{th} anomaly and the n^{th} (multiplicative) central charge of $\mathcal{C}$ as

$$\alpha_n(\mathcal{C}) := \frac{\tau_n(\mathcal{C})}{\tau_{-n}(\mathcal{C})} \quad \textit{and} \quad \xi_n(\mathcal{C}) := \frac{\tau_n(\mathcal{C})}{|\tau_n(\mathcal{C})|}.$$

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Generalizations of the quadratic Gauss sum and the Jacobi symbol.

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Consider $(\mathbb{Z}/p\mathbb{Z}, q_a)$ such that $q_a(1) = e^{2\pi i a/p}$ for $p \nmid a$. For $\mathcal{C}_a := \mathcal{C}(\mathbb{Z}/p\mathbb{Z}, q_a)$,

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$$\tau_1(\mathcal{C}_a) = \left(\frac{a}{p}\right)\sqrt{\left(\frac{-1}{p}\right)p} \quad \xi_1(\mathcal{C}_a) = \left(\frac{a}{p}\right)\sqrt{\left(\frac{-1}{p}\right)}$$

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$\tau_1(\mathcal{C}_{an}) = \tau_n(\mathcal{C}_a)$, and $\xi_1(\mathcal{C}_{an}) = \xi_n(\mathcal{C}_a)$. Moreover, they are Galois conjugates to each other.

Properties of $\tau_{\pm 1}$

- $\tau_1 \tau_{-1} = |\tau_1|^2 = \dim(\mathcal{C})$.
- ξ_1 is a root of unity.
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Example. Consider $\mathcal{C}(\mathbb{Z}/2\mathbb{Z}, q)$ where $q(1) = 1, q(-1) = i$. $\tau_2(\mathcal{C}(\mathbb{Z}/2\mathbb{Z}, q)) = 0$.

Higher Gauss sums

Example. The modular data of the modular categories $\mathcal{C}(\mathfrak{g}, k)$ can be extracted from the Kac-Peterson formula. For $\mathcal{C}(\mathfrak{g}_2, 3)$, one computes

$$\xi_3(\mathcal{C}) = \frac{1}{2\sqrt{2}}(\sqrt{7} + i),$$

which has minimal polynomial $2x^4 - 3x^2 + 2$ and thus is not a root of unity.

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Example. (Shimizu) There exists a 27-dimensional Hopf algebra H associated to a primitive 3^{rd} root of unity ζ_3. Let $\mathcal{C} = \mathcal{Z}(\mathrm{Rep}(H))$, we have

$$\alpha_3(\mathcal{C}) = \frac{5 + 4\zeta_3^2}{5 + 4\zeta_3}$$

is $7x^2 + 2x + 7$, α_3 and ξ_3 are not roots of unity. Note that $\dim(\mathcal{C})$ is a power of 3.

Arithmetic properties

Let $\mathcal{C}$ be modular, $s = S/\sqrt{\dim(\mathcal{C})}$. For any cube root γ of ξ_1, let $t = \gamma^{-1}T$.
For any $Y \in \mathrm{Irr}(\mathcal{C})$, $\chi_Y : K_0 \to \mathbb{C}$, $\chi_Y(X) = \frac{s_{X,Y}}{s_{\mathbb{1},Y}}$ is a character ($\mathbb{C}$-algebra homomorphism) of K_0.
S-matrix is invertible $\Rightarrow$ every character is given by some χ_Y.
For all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $\sigma \circ \chi_Y$ is another character. Therefore, σ induces a permutation $\hat{\sigma} : \mathrm{Irr}(\mathcal{C}) \to \mathrm{Irr}(\mathcal{C})$ such that

$$\sigma\left(\frac{s_{X,Y}}{s_{\mathbb{1},Y}}\right) = \sigma(\chi_Y(X)) = \chi_{\hat{\sigma}(Y)}(X) = \frac{s_{X,\hat{\sigma}(Y)}}{s_{\mathbb{1},\hat{\sigma}(Y)}}$$

and

$$\sigma^2(t_X) = t_{\hat{\sigma}(X)}.$$

Arithmetic properties

Theorem (Ng-Schopieray-W.)

Let $\mathcal{C}$ be a modular category, $N = \mathrm{ord}(T_{\mathcal{C}})$, and $n \in \mathbb{Z}$ relatively prime to N. Then, for $a \in \mathbb{Z}$,

$$\tau_{an}(\mathcal{C}) = \sigma(\tau_a(\mathcal{C}))\frac{\dim(\mathcal{C})}{\sigma(\dim(\mathcal{C}))}\theta^{an}_{\hat{\sigma}(\mathbb{1})} \quad \text{and} \tag{1}$$

where $\tilde{n}$ is the multiplicative inverse of n modulo N, and σ is any automorphism of $\overline{\mathbb{Q}}$ such that $\sigma(\zeta_N) = \zeta_N^{\tilde{n}}$. In particular, $\tau_n(\mathcal{C}) \neq 0$ and $\xi_n(\mathcal{C})$ is well-defined.

An algebraic integer $a \in \mathbb{C}$ is called a d-number if the principal ideal (a) in the ring of integers of $\bar{\mathbb{Q}}$ is fixed by $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. For example, $\sqrt{5}$ is a d-number.

Corollary

With the same notations as above, we have

(a) *the n^{th} anomaly of $\mathcal{C}$ is given by*

$$\alpha_n(\mathcal{C}) = \sigma(\alpha_1(\mathcal{C})) \cdot \theta^{2n}_{\hat{\sigma}(\mathbb{1})}\,. \tag{2}$$

In particular, $\alpha_n(\mathcal{C})$ and $\xi_n(\mathcal{C})$ are both roots of unity.

(b) *The n^{th} Gauss sum of $\mathcal{C}$ is a d-number.*

Example. $\tau_1(\mathcal{C}(\mathbb{Z}/5\mathbb{Z}, q_1)) = \sum_{j=0}^{4} e^{2\pi i j^2/5} = \sqrt{5}$.

Witt invariance

Two modular categories $\mathcal{C}$ and $\mathcal{D}$ are Witt equivalent if there exist a fusion category $\mathcal{A}$ such that $\mathcal{C} \otimes \mathcal{D}^{\mathrm{rev}} \cong \mathcal{Z}(\mathcal{A})$ as braided fusion categories.
Generalization of the classical Witt group $\mathcal{W}_{\mathrm{pt}}$ of quadratic modules.

Theorem (Ng-Schopieray-W.)

Let $\mathcal{C}$ and $\mathcal{D}$ be Witt equivalent psuedounitary modular categories. For $n \in \mathbb{Z}$ coprime to $\operatorname{ord}(T_{\mathcal{C}}) \operatorname{ord}(T_{\mathcal{D}})$, we have

$$\xi_n(\mathcal{C}) = \xi_n(\mathcal{D}).$$

Witt invariance

Example. Let $\mathcal{C} = \mathcal{C}(\mathbb{Z}/4\mathbb{Z}, q)$ and $\mathcal{D} = \mathcal{C}(\mathbb{Z}/4\mathbb{Z}, q) \boxtimes \mathcal{C}(\mathbb{Z}/2\mathbb{Z}, q')$ where $q(1) = \zeta_8$, $q'(1) = -i$. Let $\mathcal{E}_{a,b} = \mathcal{C}^{\boxtimes a} \boxtimes \mathcal{D}^{\boxtimes b}$, then

$$\xi_1(\mathcal{E}_{a,b}) = \zeta_8^a, \quad \xi_3(\mathcal{E}_{a,b}) = (-1)^b \zeta_8^{3a}.$$

The subgroup of $\mathcal{W}_{\mathrm{pt}}(2)$ of 2-groups generated by $[\mathcal{C}]$ and $[\mathcal{D}]$ gives 16 pairs of higher central charges (ξ_1, ξ_3).
Since $\mathcal{W}_{\mathrm{pt}}(2)$ is of order 16, $[\mathcal{C}]$ and $[\mathcal{D}]$ are generators of $\mathcal{W}_{\mathrm{pt}}(2)$.

Thank You!